

EQUATION OF STATE IN QUANTUM CHROMODYNAMICS

V. I. Yukalov^{1†} and E. P. Yukalova²

(1) *Bogoliubov Laboratory of Theoretical Physics*

Joint Institute for Nuclear Research, Dubna 141980, Russia

(2) *Laboratory of Computing Techniques and Automation*

Joint Institute for Nuclear Research, Dubna 141980, Russia

† *E-mail:* yukalov@thsun1.jinr.ru

Abstract

The equation of state for quark–gluon plasma is obtained, being valid for arbitrary values of the coupling parameter and of temperature. This equation is constructed from perturbative expansions for free energy and for renormalization function. Summation of asymptotic perturbative series is accomplished by means of the self–similar approximation theory.

Key-words: Quark–gluon plasma, equation of state, confinement, self–similar approximants

1 Introduction

Under an equation of state one implies the dependence of pressure or free energy on some characteristic parameters, for instance, coupling parameter or temperature. The definition of pressure is

$$p = \frac{T}{V} \ln \text{Tr } e^{-H/T},$$

where T is temperature; V , volume; and H is the Hamiltonian defining the considered system. In recent years there has been much attention paid to understanding the equation of state in quantum chromodynamics. The state of matter, called quark–gluon plasma, almost certainly existed in the early Universe up to 10^{-5} s after the Big Bang and is likely to be found in the interior of neutron stars. One hopes that this state can be formed in the collisions of relativistic heavy nuclei where it can be studied in a controlled and systematic way [1–6]. The formation of quark droplets in colliding nuclei can occur at already existing accelerator energies, leading to the cumulative effect [7].

There are two ways of obtaining the equation of state in quantum chromodynamics. One is a numerical way based on lattice simulation, and another way is by employing statistical models. A discussion of these ways can be found in the recent review [6]. In this report we advance the third possibility of deriving the equation of state in quantum chromodynamics. This novel way is based on the summation of perturbative series in powers of the coupling parameter. The method of summation we use here is a variant of the self–similar approximation theory [8–15] employing self–similar exponential approximants [16]. This method has been successfully used for constructing accurate equations of state for several physical systems [15,16].

2 Summation of series

Perturbation theory can be used in *QCD* at high temperatures when the coupling parameter becomes small. One usually considers the case of zero chemical potential and assumes that the temperature is high enough that fermion masses can be ignored. An expansion for the *QCD* free energy or pressure, in powers of the coupling parameter g has been known to fourth order [17–20] and recently was calculated to fifth order [21,22].

The perturbative expansion for the pressure of high-temperature gauge theory with massless fermions in four dimensions can be written [21] as

$$p(g) \simeq \frac{\pi^2 d_A}{45} T^4 \sum_n (c_n + c'_n \ln g) g^n , \quad (1)$$

where the coefficient are

$$\begin{aligned} c_0 &= 1 + \frac{7d_F}{4d_A} , \quad c_1 = 0 , \quad c_2 = -\frac{5}{(4\pi)^2} \left(C_A + \frac{5}{2} S_F \right) , \quad c_3 = \frac{240}{(4\pi)^3} \left(\frac{C_A + S_F}{3} \right)^{3/2} , \\ c_4 &= -\frac{5}{(4\pi)^4} \left[C_A^2 \left(\frac{38\zeta'(-3)}{3\zeta(-3)} - \frac{148\zeta'(-1)}{3\zeta(-1)} + \frac{64}{5} - 4\gamma_E + \frac{22}{3} \ln \frac{\mu}{4\pi T} \right) + \right. \\ &\quad + C_A S_F \left(\frac{\zeta'(-3)}{3\zeta(-3)} - \frac{74\zeta'(-1)}{3\zeta(-1)} + \frac{1759}{60} + \frac{37}{5} \ln 2 - 8\gamma_E + \frac{47}{3} \ln \frac{\mu}{4\pi T} \right) + \\ &\quad + S_F^2 \left(\frac{8\zeta'(-3)}{3\zeta(-3)} - \frac{16\zeta'(-1)}{3\zeta(-1)} - \frac{1}{3} + \frac{88}{5} \ln 2 - 4\gamma_E - \frac{20}{3} \ln \frac{\mu}{4\pi T} \right) + \\ &\quad \left. + S_{2F} \left(24 \ln 2 - \frac{105}{4} \right) - 48C_A(C_A + S_F) \ln \left(\frac{1}{2\pi} \sqrt{\frac{C_A + S_F}{3}} \right) \right] , \\ c_5 &= \frac{5}{(4\pi)^5} \sqrt{\frac{C_A + S_F}{3}} \left[C_A^2 \left(264 \ln 2 - 494 - 24\pi^2 + 176\gamma_E + 176 \ln \frac{\mu}{4\pi T} \right) + \right. \\ &\quad + C_A S_F \left(72 - 128 \ln 2 + 112\gamma_E + 112 \ln \frac{\mu}{4\pi T} \right) + \\ &\quad \left. + S_F^2 \left(32 - 128 \ln 2 - 64\gamma_E - 64 \ln \frac{\mu}{4\pi T} \right) - 144S_{2F} \right] , \\ c'_0 &= c'_1 = c'_2 = c'_3 = c'_5 = 0 , \quad c'_4 = \frac{240}{(4\pi)^4} C_A(C_A + S_F) . \end{aligned}$$

Here the dimensionless regularization is used, the scale μ corresponds to the modified minimal subtraction scheme, $\zeta(\cdot)$ is the Riemann zeta function, γ_E is the Euler–Mascheroni constant, and for $SU(N_c)$ theory, with N_c colours and with n_f fermions in the fundamental representation,

$$d_A = N_c^2 - 1 , \quad d_F = N_c n_f , \quad C_A = N_c ,$$

$$S_F = \frac{1}{2} n_f , \quad S_{2F} = \frac{N_c^2 - 1}{4N_c} n_f .$$

For *QCD* with $N_c = 3$, one has

$$d_A = 8 \quad d_F = 3n_f , \quad C_A = 3 , \quad S_F = \frac{1}{2} n_f , \quad S_{2F} = \frac{2}{3} n_f .$$

Then expansion (1) can be written as

$$p(g) \simeq \frac{8\pi^2}{45} T^4 \left(\sum_n c_n g^n + \ln g \sum_n c'_n g^n \right), \quad (2)$$

with the nonzero coefficients

$$\begin{aligned} c_0 &= 1 + \frac{21}{32} n_f, & c_2 &= -0.09499 \left(1 + \frac{5}{12} n_f \right), & c_3 &= 0.12094 \left(1 + \frac{1}{6} n_f \right)^{3/2}, \\ c_4 &= 0.04331 \left(1 + \frac{1}{6} n_f \right) \ln \left(1 + \frac{1}{6} n_f \right) + \\ &+ 0.01733 - 0.00763 n_f - 0.00088 n_f^2 - 0.01323 \left(1 + \frac{5}{12} n_f \right) \left(1 - \frac{2}{33} n_f \right) \ln \frac{\mu}{T}, \\ c_5 &= - \left(1 + \frac{1}{6} n_f \right)^{1/2} \left(0.12806 + 0.00717 n_f - 0.00027 n_f^2 \right) + \\ &+ 0.02527 \left(1 + \frac{1}{6} n_f \right)^{3/2} \left(1 - \frac{2}{33} n_f \right) \ln \frac{\mu}{T}, \\ c'_4 &= 0.08662 \left(1 + \frac{1}{6} n_f \right). \end{aligned}$$

It is convenient to introduce the dimensionless function

$$f(g) \equiv \frac{p(g)}{p(0)}, \quad p(0) = \frac{8\pi^2}{45} \left(1 + \frac{21}{32} n_f \right) T^4, \quad (3)$$

being pressure (2) reduced to the Stefan–Boltzmann limit $p(0)$. As $g \rightarrow 0$, the reduced pressure (3) reads

$$f(g) \simeq \sum_{n=0} a_n g^n + \ln g \sum_{n=2} a'_n g^{2n}, \quad (4)$$

where $a_n \equiv c_n/c_0$ and $a'_n \equiv c'_n/c_0$. Truncating the asymptotic series (4), we have the sequence of perturbative approximations

$$f_k(g) = 1 + a_2 g^2 + \dots + a_k g^k, \quad (5)$$

for $k = 2, 3, 4, 5$, and

$$f_6(g) = 1 + a_2 g^2 + \dots + a_6 g^6 + \ln g (a'_4 g^4 + a'_6 g^6) \quad (6)$$

for $k = 6$. Note that the term $g^4 \ln g$ has to be ascribed to the sixth-order approximation since only then one gets the series factoring $\ln g$, which could be summed.

The renormalization scale μ can be considered as a control function which can be defined from a fixed-point condition, e.g. from the minimal-difference condition [23]

$$f_k(g, \mu) - f_{k-1}(g, \mu) = 0, \quad (7)$$

applied to the first k approximation when the scale μ appears. In our case, this is $k = 4$, for which condition (7) is equivalent to $a_4 = 0$ or $c_4 = 0$. Then we find

$$\ln \frac{\mu}{T} \equiv \ln \gamma = \frac{0.04331 \left(1 + \frac{1}{6} n_f \right) \ln \left(1 + \frac{1}{6} n_f \right) + 0.01733 - 0.00763 n_f - 0.00088 n_f^2}{0.01323 \left(1 + \frac{5}{12} n_f \right) \left(1 - \frac{2}{33} n_f \right)}. \quad (8)$$

This gives $\mu \sim T$, which is physically reasonable since T is the natural typical scale for the quark-gluon plasma. For example, $\gamma = 3.70580$ if $n_f = 0$; $\gamma = 1.69564$ is $n_f = 3$, and $\gamma = 0.99696$ if $n_f = 6$. Note that if one would take $k = 5$ in condition (7), that is $a_4 = c_4 = 0$, one would get a physically unreasonable value $\mu \sim 100T$.

Following the method of self-similar exponential approximants [15,16], starting from the sequence $\{f_k(g)\}$ with $f_k(g)$ given in (5), we construct the exponentials

$$\begin{aligned} F_2(g, \tau) &= \exp(b_2 g^2 \tau) , \\ F_3(g, \tau) &= \exp(b_2 g^2 \exp(b_3 g \tau)) , \\ F_5(g, \tau) &= \exp(b_2 g^2 \exp(b_3 g \exp(b_5 g^2 \tau))) , \end{aligned} \quad (9)$$

in which $F_4 = F_3$ because of (7), the coefficients are

$$b_2 = a_2 = \frac{c_2}{c_0} , \quad b_3 = \frac{a_3}{a_2} = \frac{c_3}{c_2} , \quad b_5 = \frac{a_5}{a_3} = \frac{c_5}{c_3} ,$$

and $\tau = \tau_k(g)$ is a control function for the corresponding expression $F_k(g, \tau)$, so that

$$F_k(g, \tau) - F_{k-1}(g, \tau) = 0 , \quad \tau = \tau_k(g) ,$$

similarly to (7), with $\tau_2(g) \equiv 1$.

After defining the control functions $\tau_k(g)$, we obtain the self-similar approximants

$$f_k^*(g) \equiv F_k(g, \tau_k(g)) . \quad (10)$$

In particular,

$$\begin{aligned} f_2^*(g) &= \exp(b_2 g^2) , \\ f_3^*(g) &= \exp(b_2 g^2 \tau_3(g)) , \\ f_5^*(g) &= \exp(b_2 g^2 \exp(b_3 g \tau_5(g))) , \end{aligned} \quad (11)$$

with the control functions defined by the equations

$$\tau_3 = \exp(b_3 g \tau_3) , \quad \tau_5 = \exp(b_5 g^2 \tau_5) .$$

In expansion (4) and, respectively, in approximants (11), the coupling parameter $g = g(\mu)$ is the running coupling described by the renormalization group equation

$$\mu \frac{\partial g}{\partial \mu} = \beta(g) . \quad (12)$$

The renormalization function $\beta(g)$, as $g \rightarrow 0$, has the expansion [24]

$$\beta(g) \simeq \beta_3 g^3 + \beta_5 g^5 + \beta_7 g^7 , \quad (13)$$

in which

$$\begin{aligned} \beta_3 &= -\frac{1}{(4\pi)^2} \left(11 - \frac{2}{3} n_f \right) , \quad \beta_5 = -\frac{2}{(4\pi)^4} \left(51 - \frac{19}{3} n_f \right) , \\ \beta_7 &= -\frac{1}{2(4\pi)^6} \left(2857 - \frac{5033}{9} n_f + \frac{325}{27} n_f^2 \right) . \end{aligned}$$

As the self-similar approximants for (13), we have

$$\begin{aligned}\beta_5^*(g) &= ag^3 \exp(bg^2), \\ \beta_7^*(g) &= ag^3 \exp\left(bg^2\tau_7(g)\right),\end{aligned}\quad (14)$$

where the control function $\tau_7(g)$ is the solution of the equation $\tau_7 = \exp(cg^2\tau_7)$ and the coefficients are $a \equiv \beta_3$, $b \equiv \beta_5/\beta_3$, and $c \equiv \beta_7/\beta_5$.

To solve equation (12) with the right-hand side given by one of the functions from (14), we need an initial condition. For the latter, we may accept the value $\alpha_s(m_Z) = 0.119$ of the strong coupling constant $\alpha_s = g^2/4\pi$ at the Z^0 boson mass m_Z [25]. Thus, the initial condition for equation (12) is

$$g(m_Z) = 1.222856, \quad m_Z = 91.187 \text{ GeV}.$$

Solving (12), we find $g = g(\mu)$. Substituting the latter in (10) and using the relation $\mu = \gamma T$ following from (8), we obtain the reduced pressure

$$\bar{f}_k(T) \equiv f_k^*(g(\gamma T)) \quad (15)$$

as a function of temperature.

3 Results and Conclusion

The results of our calculations for $n_f = 6$ are presented in Figs. 1 to 3. The self-similar approximants $f_k^*(g)$ given by (11) for the reduced pressure (3), as functions of the coupling parameter g , are shown in Fig. 1. The solution $g(\mu)$ to the evolution equation (12) is drawn in Fig. 2. Both functions in (14) give practically the same solution $g(\mu)$. Figure 3 presents the equation of state (15).

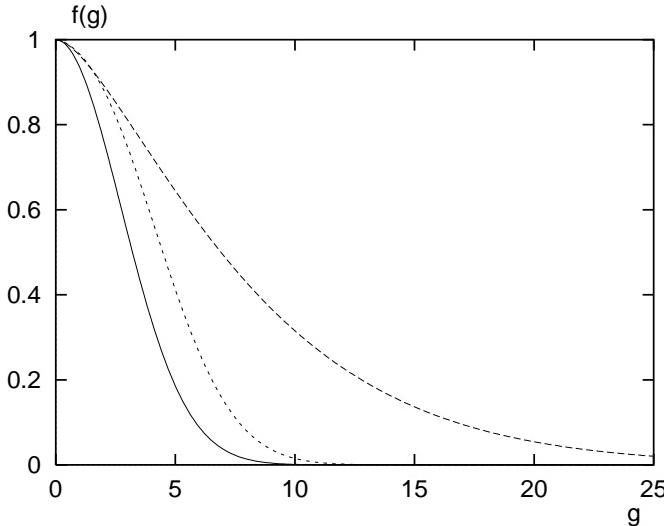


Figure 1: The behaviour of the reduced pressure $f(g) \equiv p(g)/p(0)$ presented by the self-similar approximations $f_2^*(g)$ (solid line), $f_3^*(g)$ (long-dashed line), and $f_5^*(g)$ (short-dashed line).

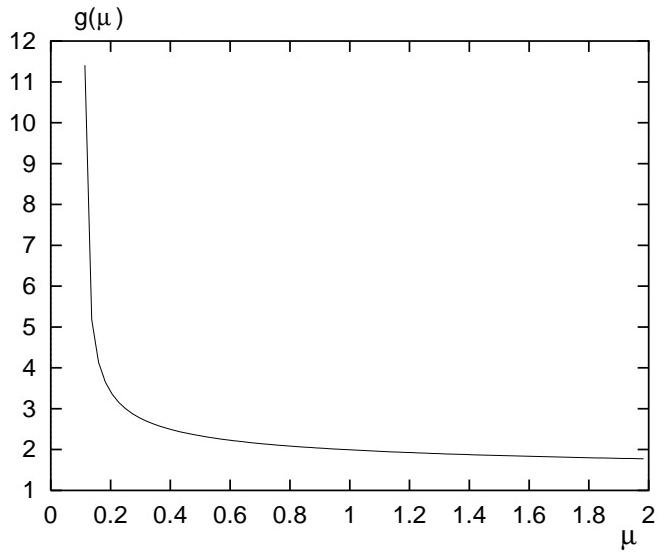


Figure 2: The running coupling $g(\mu)$ as a function of the scale μ , obtained from the renormalization-group equation (12).

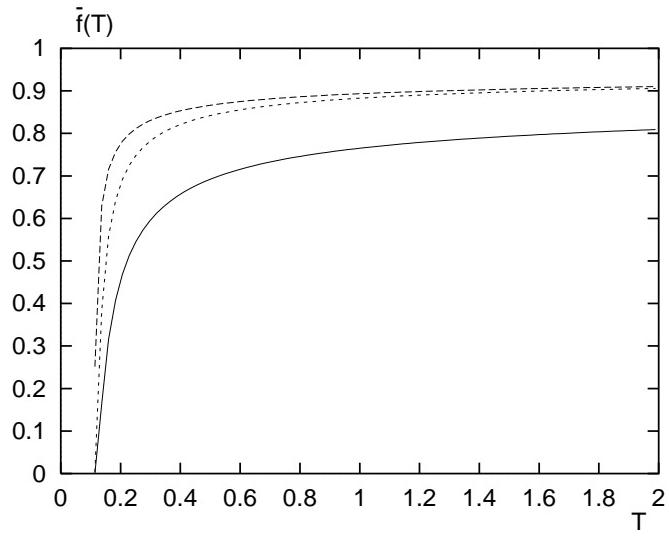


Figure 3: The reduced pressure $\bar{f}(T)$ given by the self-similar approximants $\bar{f}_2(T)$ (solid line), $\bar{f}_3(T)$ (long-dashed line), and $\bar{f}_5(T)$ (short-dashed line).

The overall behaviour of the reduced pressure $\bar{f}(T)$ is in agreement with that found in lattice simulations and in statistical modelling (see review [6]). At the temperature $T_c \approx (150 - 200)\text{MeV}$ the pressure sharply drops down to practically zero, which can be interpreted as confinement. The curves corresponding to subsequent approximations $k = 2, 3, 5$ are close to each other. The qualitative behaviour of $\bar{f}(T)$ for different numbers of flavours in the diapason $0 \leq n_f \leq 8$ is the same, with increasing T_c as n_f decreases. Certainly, in the vicinity of $T_c \sim 200\text{MeV}$ one should take into account the quark masses, especially those of heavy quarks. But what is very interesting is that, starting from perturbative expansions in the high-temperature region of asymptotic freedom, we obtained an equation of state with the qualitatively correct behaviour, at all temperatures, including the existence of confinement at $T_c \sim 200\text{MeV}$.

References

- [1] H. Halzen, *Contemp. Phys.* **24**, 591 (1983).
- [2] J. Cleymans, R.V. Gavai, and E. Suhonen, *Phys. Rep.* **130**, 217 (1986).
- [3] R.B. Clare and D. Strottman, *Phys. Rep.* **141**, 177 (1986).
- [4] H. Reeves, *Phys. Rep.* **201**, 335 (1991).
- [5] C. Adami and G.E. Brown, *Phys. Rep.* **234**, 1 (1993).
- [6] V.I. Yukalov and E.P. Yukalova, *Phys. Part. Nucl.* **28**, 37 (1997).
- [7] A.M. Baldin, *Phys. Part. Nucl.* **8**, 429 (1977).
- [8] V.I. Yukalov, *Physica A* **167**, 833 (1990).
- [9] V.I. Yukalov, *Phys. Rev. A* **42**, 3324 (1990).
- [10] V.I. Yukalov, *J. Math. Phys.* **32**, 1235 (1991).
- [11] V.I. Yukalov, *J. Math. Phys.* **33**, 3994 (1992).
- [12] V.I. Yukalov and E.P. Yukalova, *Physica A* **225**, 336 (1996).
- [13] V.I. Yukalov and S. Gluzman, *Phys. Rev. Lett.* **79**, 333 (1997).
- [14] S. Gluzman and V.I. Yukalov, *Phys. Rev. E* **55**, 3983 (1997).
- [15] V.I. Yukalov and S. Gluzman, *Phys. Rev. E* **55**, 6552 (1997).
- [16] V.I. Yukalov and S. Gluzman, *Phys. Rev. E* **58**, 1359 (1998).
- [17] J.I. Kapusta, *Nucl. Phys. B* **148**, 461 (1979).
- [18] E.V. Shuryak, *Phys. Rep.* **61**, 71 (1980).
- [19] P. Arnold and C. Zhai, *Phys. Rev. D* **50**, 7603 (1994).

- [20] P. Arnold and C. Zhai, Phys. Rev. D **51**, 1906 (1995).
- [21] C. Zhai and B. Kastening, Phys. Rev. D **52**, 7232 (1995).
- [22] E. Braaten and A. Nieto, Phys. Rev. D **53**, 3421 (1996).
- [23] V.I. Yukalov, Mosc. Univ. Phys. Bull. **31**, 10 (1976).
- [24] T. van Ritbergen, J.A.M. Vermaseren, and S.A. Larin, Phys. Lett. B **400**, 379 (1997).
- [25] Review of Particle Physics, Eur. Phys. J. C **3**, 1 (1998).